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The soliton solutions of the (1 + 1)-dimensional real $\phi^3+\phi^4$ field at finite temperature

Ru-keng Su†‡ and Xiang-guang Gu§||

† Physics Department, Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, NY 11794, USA
§ Physics Department, Fudan University, Shanghai, People's Republic of China

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Abstract. By means of the method of coherent states and the real time Green function, the spontaneous breaking of symmetry in the $(1+1)$-dimensional real $\phi^3+\phi^4$ field and its restoration at finite temperature are investigated. A new soliton solution which has essential differences from the kink and antikink solutions of ϕ^4 field is found. The soliton mass and the critical temperature are given.

1. Introduction

In a series of previous papers, using the generalised Bogoliubov transformation to construct a coherent state under the pairing approximation of the real time Green functions, we have studied the $(1+1)$-dimensional real Higgs ϕ^4 field (Su *et al* 1983), the $(1+3)$-dimensional real Higgs ϕ^4 field (Su 1983), the σ model (Su and Bi 1984a, b), the complex $(1+3)$-dimensional Higgs ϕ^4 field, the $U(1)+\phi^4$ field (Su and Bi 1984a) and the Mohapatra-Senjanovic model (Mohapatra and Senjanovic 1979a, b, c, 1980, Su and Bi 1984b), and given their elementary excitation spectra and corresponding critical temperatures. All these results confirm that this method is successful for degenerate symmetric Bose systems. A problem which then naturally arises is: can this method also be used to discuss the non-degenerate non-symmetric Bose system? In this paper, as an example, we shall discuss the $\phi^3+\phi^4$ field. Two different vacua—one the global (real vacuum), the other the local (false vacuum)—occur, which are distinct since the global symmetry $\phi\leftrightarrow-\phi$ is destroyed by the presence of the ϕ^3 term. This is the non-degenerate case. It can easily be seen that we cannot get the kink and antikink soliton. However, as we can prove later, we can obtain another soliton solution which corresponds to a one-dimensional motion from the turning point to the false vacuum.

The organisation of this paper is as follows. In § 2, after quantising the Hamiltonian of $\phi^3+\phi^4$, we perform the Bogoliubov transformation and find the three types of solution, one of which describes a new soliton. In § 3, using the real time Green function method at zero temperature, under the first-order pair cut-off approximation, we get the elementary excitation spectra in these three cases. In § 4, we extend these

‡ On leave from Department of Physics, Fudan University, Shanghai, People's Republic of China.
|| Present address: Physics Department, California Institute of Technology, Caltech, Pasadena, California 91125, USA.

results to finite temperature and obtain the critical temperature above which the symmetry is restored. In § 5, a summary and discussion are given.

2. Hamiltonian and soliton solutions

The Lagrangian density of the $(1+1)$-dimensional real $\phi^3+\phi^4$ model is

$$\mathscr{L}=\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2-\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+\frac{m^2}{4}\phi^2-\frac{g^2}{4}\phi^4-\frac{\lambda}{3}\phi^3 \tag{2.1}$$

where $\lambda>0$ is a small parameter and can be considered as perturbation. Here and hereafter we use the same notation as Su *et al* (1983). Performing the canonical quantisation

$$\phi(x,t)=\sum_k\frac{1}{(2L\omega_k)^{1/2}}(\hat{a}_k+\hat{a}^\dagger_{-k})\exp(\mathrm{i}kx) \tag{2.2}$$

with

$$[\hat{a}_k(t),\hat{a}^\dagger_{k'}(t)]=\delta_{kk'} \tag{2.3}$$

and

$$\omega_k=(\mu^2+k^2)^{1/2} \tag{2.4}$$

we obtain the Hamiltonian of our system as

$$:H:=H_2+H_3+H_4 \tag{2.5}$$

$$H_2=\sum_k\left[\left(\frac{k^2}{2\omega_k}+\frac{\omega_k}{2}-\frac{m^2}{4\omega_k}\right)\hat{a}^\dagger_k\hat{a}_k+\left(\frac{k^2}{4\omega_k}-\frac{\omega_k}{4}-\frac{m^2}{8\omega_k}\right)(\hat{a}^\dagger_k\hat{a}^\dagger_{-k}+\hat{a}_k\hat{a}_{-k})\right] \tag{2.6}$$

$$\begin{aligned}H_3=\frac{\lambda}{3}\sum_{k_1k_2k_3}\frac{1}{2(2L)^{1/2}}\frac{\delta_{k_1+k_2+k_3,0}}{(\omega_{k_1}\omega_{k_2}\omega_{k_3})^{1/2}}[&\hat{a}_{k_1}\hat{a}_{k_2}\hat{a}_{k_3}+\hat{a}^\dagger_{-k_1}\hat{a}^\dagger_{-k_2}\hat{a}^\dagger_{-k_3}\\&+3(\hat{a}^\dagger_{-k_1}\hat{a}_{k_2}\hat{a}_{k_3}+\hat{a}^\dagger_{-k_1}\hat{a}^\dagger_{-k_2}\hat{a}_{k_3})]\end{aligned} \tag{2.7}$$

$$\begin{aligned}H_4=\frac{g^2}{16L}\sum_{k_1k_2k_3k_4}\frac{\delta_{k_1+k_2+k_3+k_4,0}}{(\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_{k_4})^{1/2}}[&\hat{a}_{k_1}\hat{a}_{k_2}\hat{a}_{k_3}\hat{a}_{k_4}+\hat{a}^\dagger_{-k_1}\hat{a}^\dagger_{-k_2}\hat{a}^\dagger_{-k_3}\hat{a}^\dagger_{-k_4}\\&+4(\hat{a}^\dagger_{-k_1}\hat{a}_{k_2}\hat{a}_{k_3}\hat{a}_{k_4}+\hat{a}^\dagger_{-k_1}\hat{a}^\dagger_{-k_2}\hat{a}^\dagger_{-k_3}\hat{a}_{k_4})+6\hat{a}^\dagger_{-k_1}\hat{a}^\dagger_{-k_2}\hat{a}_{k_3}\hat{a}_{k_4}].\end{aligned} \tag{2.8}$$

In order to introduce the coherent state configurations, let us perform the generalised Bogoliubov transformation

$$\hat{a}(k)=f(k)+\hat{c}(k). \tag{2.9}$$

Noticing that

$$\hat{a}_k\leftrightarrow(2\pi/L)^{1/2}\hat{a}(k)$$

we can rewrite the Hamiltonian (2.5) as

$$:H:=H'_0+H'_1+H'_2+H'_3+H'_4 \tag{2.10}$$

$$H_0'=\int\left\{\left[\frac{1}{2}\left(\frac{k^2}{\omega_k}+\omega_k\right)-\frac{m^2}{4\omega_k}\right]f^*(k)f(k)\right.$$

$$\left.+\left[\frac{1}{2}\left(\frac{k^2}{2\omega_k}-\frac{\omega_k}{2}\right)-\frac{m^2}{8\omega_k}\right][f(k)f(-k)+f^*(k)f^*(-k)]\right\}\mathrm{d}k$$

$$+\frac{g^2}{32\pi}\int\frac{\delta(k_1+k_2+k_3+k_4)}{(\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_{k_4})^{1/2}}\{f(k_1)f(k_2)f(k_3)f(k_4)$$

$$+f^*(-k_1)f^*(-k_2)f^*(-k_3)f^*(-k_4)$$

$$+4[f^*(-k_1)f^*(-k_2)f^*(-k_3)f(k_4)+f^*(-k_1)f(k_2)f(k_3)f(k_4)]$$

$$+6f^*(-k_1)f^*(-k_2)f(k_3)f(k_4)\}\ \mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}k_4$$

$$+\frac{\lambda}{12\sqrt{\pi}}\int\frac{\delta(k_1+k_2+k_3)}{(\omega_{k_1}\omega_{k_2}\omega_{k_3})^{1/2}}\{f(k_1)f(k_2)f(k_3)+f^*(-k_1)f^*(-k_2)f^*(-k_3)$$

$$+3[f^*(-k_1)f(k_2)f(k_3)+f^*(-k_1)f^*(-k_2)f(k_3)]\}\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3 \qquad (2.11)$$

$$H_1'=\int\left[\frac{1}{2}\left(\frac{k^2}{\omega_k}+\omega_k\right)-\frac{m^2}{4\omega_k}\right][f(-k)\hat{c}^\dagger(-k)+f^*(k)\hat{c}(k)]\,\mathrm{d}k$$

$$+\int\left[\frac{1}{2}\left(\frac{k^2}{\omega_k}-\omega_k\right)-\frac{m^2}{4\omega_k}\right][f(-k)\hat{c}(k)+f^*(k)\hat{c}^\dagger(-k)]\ \mathrm{d}k$$

$$+\frac{g^2}{8\pi}\int\frac{\delta(k_1+k_2+k_3+k_4)}{(\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_{k_4})^{1/2}}\{[f(k_1)f(k_2)f(k_3)+f^*(-k_1)f^*(-k_2)f^*(-k_3)$$

$$+3[f^*(-k_1)f(k_2)f(k_3)$$

$$+f^*(-k_1)f^*(-k_2)f(k_3)]][\hat{c}(k_4)+\hat{c}^\dagger(-k_4)]\}\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}k_4$$

$$+\frac{\lambda}{12\sqrt{\pi}}\int\frac{\delta(k_1+k_2+k_3)}{(\omega_{k_1}\omega_{k_2}\omega_{k_3})^{1/2}}\{[3f(k_1)f(k_2)+6f^*(-k_1)f(k_2)$$

$$+3f^*(-k_1)f^*(-k_2)][\hat{c}(k_3)+\hat{c}^\dagger(-k_3)]\}\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3 \qquad (2.12)$$

$$H_2'=\int\left\{\left[\frac{1}{2}\left(\frac{k^2}{\omega_k}+\omega_k\right)-\frac{m^2}{4\omega_k}\right]\hat{c}^\dagger(k)\hat{c}(k)+\left[\frac{1}{2}\left(\frac{k^2}{\omega_k}-\frac{\omega_k}{2}\right)-\frac{m^2}{8\omega_k}\right]\right.$$

$$\left.\times[\hat{c}(k)\hat{c}(-k)+\hat{c}^\dagger(k)\hat{c}^\dagger(-k)]\right\}\mathrm{d}k+\frac{3g^2}{16\pi}\int\frac{\delta(k_1+k_2+k_3+k_4)}{(\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_{k_4})^{1/2}}$$

$$\times\{[f(k_1)f(k_2)+2f^*(-k_1)f(k_2)+f^*(-k_1)f^*(-k_2)][\hat{c}(k_3)\hat{c}(k_4)$$

$$+\hat{c}^\dagger(-k_3)\hat{c}^\dagger(-k_4)+2\hat{c}^\dagger(-k_3)\hat{c}(k_4)]\}\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}k_4$$

$$+\frac{\lambda}{12\sqrt{\pi}}\int\frac{\delta(k_1+k_2+k_3)}{(\omega_{k_1}\omega_{k_2}\omega_{k_3})^{1/2}}\{[3f(k_1)+3f^*(-k_1)][\hat{c}(k_2)\hat{c}(k_3)$$

$$+\hat{c}^\dagger(-k_2)\hat{c}^\dagger(-k_3)+2\hat{c}^\dagger(-k_3)\hat{c}(k_2)]\}\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3 \qquad (2.13)$$

$$H_3' = \frac{g^2}{8\pi} \int \frac{\delta(k_1+k_2+k_3+k_4)}{(\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_{k_4})^{1/2}} \{[f(k_1)+f^*(-k_1)][\hat{c}(k_2)\hat{c}(k_3)\hat{c}(k_4)$$
$$+\hat{c}^\dagger(-k_2)\hat{c}^\dagger(-k_3)\hat{c}^\dagger(-k_4)+3\hat{c}^\dagger(-k_2)\hat{c}(k_3)\hat{c}(k_4)+3\hat{c}^\dagger(-k_2)\hat{c}^\dagger(-k_3)\hat{c}(k_4)]\}$$
$$\times \mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}k_4 + \frac{\lambda}{12\sqrt{\pi}} \int \frac{\delta(k_1+k_2+k_3)}{(\omega_{k_1}\omega_{k_2}\omega_{k_3})^{1/2}} [\hat{c}(k_1)\hat{c}(k_2)\hat{c}(k_3)$$
$$+\hat{c}^\dagger(-k_1)\hat{c}^\dagger(-k_2)\hat{c}^\dagger(-k_3)+3\hat{c}^\dagger(-k_1)\hat{c}(k_2)\hat{c}(k_3)$$
$$+3\hat{c}^\dagger(-k_1)\hat{c}^\dagger(-k_2)\hat{c}(k_3)]\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3 \tag{2.14}$$

$$H_4' = \frac{g^2}{32\pi} \int \frac{\delta(k_1+k_2+k_3+k_4)}{(\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_{k_4})^{1/2}} \{\hat{c}(k_1)\hat{c}(k_2)\hat{c}(k_3)\hat{c}(k_4)$$
$$+\hat{c}^\dagger(-k_1)\hat{c}^\dagger(-k_2)\hat{c}^\dagger(-k_3)\hat{c}^\dagger(-k_4)+4[\hat{c}^\dagger(-k_1)\hat{c}^\dagger(-k_2)\hat{c}^\dagger(-k_3)\hat{c}(k_4)$$
$$+\hat{c}^\dagger(-k_1)\hat{c}(k_2)\hat{c}(k_3)\hat{c}(k_4)]+6\hat{c}^\dagger(-k_1)\hat{c}^\dagger(-k_2)\hat{c}(k_3)\hat{c}(k_4)\}\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3. \tag{2.15}$$

At zero temperature, as in Su *et al* (1983), the momentum distribution function $f(k)$ which characterises the coherent state $|f\rangle$ should be determined by

$$\delta\langle :H:\rangle/\delta f(p) = \delta H_0'/\delta f(p) = 0 \tag{2.16}$$

and

$$\delta\langle :H:\rangle/\delta f^*(p) = \delta H_0'/\delta f^*(p) = 0. \tag{2.17}$$

Thus we have

$$f(-p) = f^*(p) \tag{2.18}$$

and

$$\left[\frac{1}{2}\left(\frac{p^2}{\omega_p}+\omega_p\right)-\frac{m^2}{4\omega_p}\right]f^*(p)+\left[\frac{1}{2}\left(\frac{p^2}{\omega_p}-\omega_p\right)-\frac{m^2}{4\omega_p}\right]f(-p)$$
$$+\frac{g^2}{32\pi}\int \frac{\delta(k_1+k_2+k_3+p)}{(\omega_{k_1}\omega_{k_2}\omega_{k_3}\omega_p)^{1/2}}(4f(k_1)f(k_2)f(k_3)+4f^*(-k_1)f^*(-k_2)f^*(-k_3)$$
$$+12f^*(-k_1)f(k_2)f(k_3)+12f^*(-k_1)f^*(-k_2)f(k_3))\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3$$
$$+\frac{\lambda}{12\sqrt{\pi}}\int \frac{\delta(k_1+k_2+p)}{(\omega_{k_1}\omega_{k_2}\omega_p)^{1/2}}(3f(k_1)f(k_2)+3f^*(-k_1)f^*(-k_2)$$
$$+6f^*(-k_1)f(k_2))\,\mathrm{d}k_1\,\mathrm{d}k_2 = 0. \tag{2.19}$$

Let

$$y(p) = \frac{f(p)}{\sqrt{\omega_p}} = \int_{-\infty}^{\infty} \tilde{y}(u)\exp(-\mathrm{i}pu)\,\mathrm{d}u \tag{2.20}$$

and

$$\tilde{y}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} y(p)\exp(\mathrm{i}pu)\,\mathrm{d}p. \tag{2.21}$$

Substituting (2.20) and (2.21) into (2.19), we obtain equation for $\tilde{y}(u)$:

$$\mathrm{d}^2\tilde{y}/\mathrm{d}u^2+\tfrac{1}{2}m^2\tilde{y}-4\pi g^2\tilde{y}^3-2\sqrt{\pi}\lambda\tilde{y}^2=0. \tag{2.22}$$

We can easily prove that equation (2.22) has three types of solution.

(a)
$$\tilde{y}(u)=0 \tag{2.23}$$
$$f(k)=0. \tag{2.24}$$

The expectation value of energy in a vacuum is

$$\langle :H:\rangle = U = U(a)=0. \tag{2.25}$$

This trivial solution is just the same as the ϕ^4 field; it means that our system is in an unstable normal state.

(b)
$$\tilde{y}(u)_{\frac{1}{2}}=\frac{-\lambda\sqrt{\pi}\pm(\pi\lambda^2+2\pi g^2m^2)^{1/2}}{4\pi g^2}\equiv b_{\frac{1}{2}}. \tag{2.26}$$

Thus

$$f(k)_{\frac{1}{2}}=\sqrt{\omega_0}\delta(k)\frac{-\lambda\sqrt{\pi}\pm(\pi\lambda^2+2\pi g^2m^2)^{1/2}}{2g^2} \tag{2.27}$$

$$\langle\phi\rangle_{\frac{1}{2}}=\frac{-\lambda\mp(\lambda^2+2g^2m^2)^{1/2}}{2g^2} \tag{2.28}$$

$$\begin{aligned}U(b_1)&=-\frac{m^4L}{16g^2}-\frac{\lambda^2L}{24g^6}(\lambda^2+3g^2m^2)-\frac{\lambda L}{24g^6}(\lambda^2+2g^2m^2)^{3/2}\\ U(b_2)&=-\frac{m^4L}{16g^2}-\frac{\lambda^2L}{24g^6}(\lambda^2+3g^2m^2)+\frac{\lambda L}{24g^6}(\lambda^2+2g^2m^2)^{3/2}.\end{aligned} \tag{2.29}$$

These solutions correspond to the false vacuum and true vacuum respectively in which the boson condensation with zero momentum occurs. Obviously, if $\lambda\to 0$, the solution (2.26) reduces to the ϕ^4 solution.

(c) In the appendix we get the soliton solutions of equations (2.22) as

(c1)
$$\tilde{y}=b_1+2\left(\frac{1}{f_+}-\frac{D}{2A}\right)\mathrm{e}^{\pm\tau}\left\{\left[\frac{D}{2A}+\left(\frac{1}{f_+}-\frac{D}{2A}\right)\mathrm{e}^{\pm\tau}\right]^2-B/A\right\}^{-1} \tag{2.30}$$

$$\begin{aligned}\langle\phi\rangle=2\sqrt{\pi}\Bigg[\!\!\Bigg[\,b_1&+2\left(\frac{1}{f_+}-\frac{D}{2A}\right)\exp(\pm\sqrt{A}x)\left\{\left[\frac{D}{2A}+\left(\frac{1}{f_+}-\frac{D}{2A}\right)\right.\right.\\ &\left.\left.\times\exp(\pm\sqrt{A}x)\right]^2-B/A\right\}^{-1}\Bigg]\!\!\Bigg]\end{aligned} \tag{2.31}$$

$$\begin{aligned}f(k)=\sqrt{\omega_k}\Bigg[\!\!\Bigg[\,2\pi b_1\delta(k)&+2\left\{\left[1+\frac{D}{2A}\left(\frac{A}{B}\right)^{1/2}\right]\frac{1}{R\sqrt{A}}\sum_{n=0}^{\infty}\left(-\frac{[D/2A+(B/A)^{1/2}]}{R}\right)\right.\\ &\times\frac{(n+1)}{(n+1)^2+k^2/A}\\ &\left.+\left[1-\frac{D}{2A}\left(\frac{A}{B}\right)^{1/2}\right]\frac{1}{R\sqrt{A}}\sum_{n=0}^{\infty}\left[-\frac{[D/2A-(B/A)^{1/2}]}{R}\right]^n\frac{(n+1)}{(n+1)^2+k^2/A}\right\}\Bigg]\!\!\Bigg]\end{aligned} \tag{2.32}$$

$$U(c_1)=\frac{L}{\sqrt{A}}\left(-\frac{(4\pi)^{1/2}\lambda b_1^3}{3}-g^2b_1^4\right)+\frac{6}{\sqrt{A}}(-\tfrac{2}{3}\sqrt{\pi}b_1\lambda-2g^2b_1^2)\frac{D/Bf_+-2}{(1/f_+)^2-B/A}$$

$$+\frac{2}{\sqrt{A}}(-\tfrac{2}{3}\sqrt{\pi}\lambda-4g^2b_1)$$

$$\times\left(\frac{2D/A-(1/2f_+)(1+D^2/AB)}{[(1/f_+)^2-B/A]^2}+\frac{(1/f_+)(A/B-3D^2/4B^2)}{(1/f_+)^2-B/A}\right)$$

$$-\frac{2g^2}{\sqrt{A}}\left(\frac{[4(1/f_+)^2+2D^2/A^2-4B/3A+D^3/3A^2Bf_+-14D/3Af_+]}{[(1/f_+)^2-B/A]^3}\right.$$

$$\left.+\frac{(5D^3/12AB^2f_+-5D/6Bf_+)}{[(1/f_+)^2-B/A]^2}+\frac{(5D^3/8B^3f_+-5AD/4B^2f_+)}{(1/f_+)^2-B/A}\right)$$

$$+\left[\frac{2}{\sqrt{A}}(-2\sqrt{\pi}\lambda b_1^2-4g^2b_1^3)-\frac{6}{\sqrt{A}}(-\tfrac{2}{3}\sqrt{\pi}\lambda b_1+2g^2b_1^2)\left(\frac{D}{2B}\right)\right.$$

$$\left.+\frac{2}{\sqrt{A}}(-\tfrac{2}{3}\sqrt{\pi}\lambda-4g^2b_1)\left(\frac{3D^2}{8B^2}-\frac{A}{2B}\right)-\frac{2g^2}{\sqrt{2}}\left(\frac{5AD}{8B^2}-\frac{5D^3}{16B^3}\right)\right]$$

$$\times\left(\frac{A}{B}\right)^{1/2}\ln\left(\frac{1/f_++(B/A)^{1/2}}{1/f_+-(B/A)^{1/2}}\right). \tag{2.33}$$

(c2) Changing

$$b_1=\frac{-\sqrt{\pi}\lambda+(\pi\lambda^2+2\pi g^2m^2)^{1/2}}{4\pi g}\qquad\text{to}\qquad b_2=\frac{-\sqrt{\pi}\lambda-(\pi\lambda^2+2\pi g^2m^2)^{1/2}}{4\pi g} \tag{2.33'}$$

we get the corresponding $\tilde{y}$, $\langle\phi\rangle$, $f(k)$, $U(c_2)$ respectively.

We can easily prove that the ground-state energy given by (2.33) and (2.33′) is negative. This means that these soliton solutions, which imply that the Bose condensation not only occurs at zero momentum, but also prevails over the whole range of momenta, are stable. The physical meaning of these solutions is obvious: a particle moves from f_+ towards the false vacuum and stays there permanently. The first term $2\pi b\delta(k)$ is the contribution of the false vacuum and the other term comes from the soliton movements (see the appendix).

3. The real time Green function (zero temperature)

As in Su *et al* (1983), we can use the equation of motion for a real time Green function to find the elementary excitation spectra of the $\phi^3+\phi^4$ system under the first-order pair cut-off approximation. For simplicity, let us write down the main results only.

Define the Green function

$$G_1=\langle\langle\hat{c}(k)|\hat{c}^\dagger(k)\rangle\rangle\qquad G_2=\langle\langle\hat{c}^\dagger(-k)|\hat{c}^\dagger(k)\rangle\rangle \tag{3.1}$$

in spectral representation and we get

$$G_1=\frac{1}{2\pi}\frac{E+\Omega_k}{E^2-(\Omega_k^2-\Delta_k^2)}\qquad G_2=-\frac{1}{2\pi}\frac{\Delta_k}{E^2-(\Omega_k^2-\Delta_k^2)} \tag{3.2}$$

where

$$\Omega_k=\frac{1}{2}\left(\frac{k^2}{\omega_k}+\omega_k\right)-\frac{m^2}{4\omega_k}+\frac{3g^2}{L\omega_k}\int\frac{1}{\omega_1}f(k_1)f(-k_1)\,\mathrm{d}k_1+\frac{2\sqrt{\pi}\lambda}{L}\frac{f(0)}{\sqrt{\omega_0\omega_k}} \tag{3.3}$$

$$\Delta_k=\Omega_k-\omega_k. \tag{3.4}$$

The poles of the Green function give us the elementary excitation spectra

$$E_\mathrm{p}^2=\Omega_\mathrm{p}^2-\Delta_\mathrm{p}^2. \tag{3.5}$$

In the uniform condensation phase (2.27), we obtain

$$E_\mathrm{p}^2=p^2+m^2+(1/2g^2)[\lambda^2\mp\lambda(\lambda^2+2g^2m^2)^{1/2}]. \tag{3.6}$$

When $\lambda\to 0$, this reduces to the ϕ^4 case.

In the soliton case (2.33) and (2.33′), the elementary excitation spectrum is

$$\begin{aligned}E_\mathrm{p}^2=p^2-\frac{m^2}{2}+12\pi b^2g^2+\frac{24\pi bg^2}{L\sqrt{B}}\ln\left(\frac{(R+D/2A)^2-B/A}{R^2}\right)+\frac{24\pi g^2\sqrt{A}}{LB}\\ \times\left[-2+\frac{D}{2A}\left(\frac{A}{B}\right)^{1/2}\ln\left(\frac{[D/2A-(B/A)^{1/2}+R\exp(\sqrt{A}L)][D/2A+(B/A)^{1/2}]}{[D/2A+(B/A)^{1/2}+R\exp(\sqrt{A}L)][D/2A-(B/A)^{1/2}]}\right)\right]\\ +4\sqrt{\pi}\lambda b+\frac{8\sqrt{\pi}\lambda}{L\sqrt{B}}\ln\left(\frac{(R+D/2A)^2-B/A}{R^2}\right).\end{aligned} \tag{3.7}$$

When $L\to\infty$, it has the same value as (3.6). So in either case we find that the mass of the elementary excitation is equal to

$$\{m^2+(1/2\pi g^2)[\lambda^2\mp\lambda(\lambda^2+2g^2m^2)^{1/2}]\}^{1/2}\qquad \text{as } L\to\infty.$$

4. The critical temperature and phase transition

In Su *et al* (1983), after replacing the vacuum average by the ensemble average, the authors gave a very important result for the ϕ^4 system: all formulae at zero temperature are still valid at finite temperature except the substitution of M for m, where

$$M^2=m^2-12g^2\nu \tag{4.1}$$

and

$$\begin{aligned}\nu&=\frac{1}{2\pi}\int_0^\infty\frac{\mathrm{d}k}{\omega_k}\langle\hat{c}^\dagger(k)\hat{c}(k)+\hat{c}^\dagger(k)\hat{c}^\dagger(-k)\rangle\\ &=\frac{1}{2\pi}\int_0^\infty\frac{\mathrm{d}k}{(k^2+M^2)^{1/2}\exp[(k^2+M^2)^{1/2}/T-1]}.\end{aligned} \tag{4.2}$$

In this paper we would like to point out that this result is also valid in the $\phi^3+\phi^4$ system. Then, when $L\to\infty$, the elementary excitation spectrum in case (b) or (c) is

$$E_\mathrm{p}^2=p^2+M^2+(1/2g^2)[\lambda^2\mp\lambda(\lambda^2+2g^2M^2)^{1/2}]. \tag{4.3}$$

Following Su *et al* (1983), the equation which determines the function $M(T)$ in the weak coupling condition is

$$(1-u)\omega^3-\{m_0^2-(3g^2/\pi)[\ln(m_0/4\pi T)+\gamma]\}\omega+3g^2T=0 \tag{4.4}$$

where

$$\omega^2 = M^2 + U_{\frac{1}{2}} \tag{4.5}$$

$$U_{\frac{1}{2}} = \frac{\lambda^2}{2g^2}\left[1 \mp \left(1 + \frac{2g^2M^2}{\lambda^2}\right)^{1/2}\right] \approx \frac{\lambda^2}{2g^2}\left[1 \mp \left(1 + \frac{2g^2m^2}{\lambda^2}\right)^{1/2}\right]. \tag{4.6}$$

The critical temperature is

$$T_c \approx (2m_0^3/9\sqrt{3}g^2)(1 - U_{\frac{1}{2}})\{1 - (9g^2/2\pi m_0^2)[\ln(g^2/m_0^2) + 0.0996]\}. \tag{4.7}$$

5. Summary and discussion

Since the pioneering work of Dolan and Jackiw (1974), Bernard (1974) and Weinberg (1974) in finite temperature quantum field theory, many studies have been made of the non-degenerate scalar $\phi^3 + \phi^4$ Bose system (for example, see Aoyama and Quinn 1984). However, to our knowledge, except for the one-dimensional classical solid chain case (Sun and Lee 1979), the soliton solution (2.33) and (2.33′) has not been given in quantum field theory up to now. Obviously, if $\lambda \neq 0$, the kink and antikink specific solution cannot occur in the $\phi^3 + \phi^4$ system, since the ϕ^3 term destroys the symmetry and causes the two vacua to have different energy. However, we can find another specific soliton solution (2.33) and (2.33′) of the non-linear equation (2.22). This solution $\tilde{y}(u)$ is an even function of u, so it can satisfy the periodic boundary condition, but it cannot reduce to the kink–antikink solution when $\lambda \to 0$. It means that it is a new specific solution which is different from the kink and antikink. Notice that many authors try to use the $\phi^3 + \phi^4$ to construct a soliton bag (Goldflam and Wilets 1982, Fiebig and Hadjimichael 1984, Bi *et al* 1986). Our soliton solutions may be of interest and helpful in the solution of the bag model.

Finally, we would like to point out that the real time Green function approach with the coherent state method is also a good method for the non-degenerate Bose system, as we have seen in the $\phi^3 + \phi^4$ model.

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Appendix

In order to find the soliton solution of equation (2.22), let us discuss this equation carefully. If we imagine $\tilde{y}$ as displacement and u as time, then $d^2\tilde{y}/du^2$ is acceleration and equation (2.22) represents a particle moving in a potential (figure 1)

$$\tilde{U} = \tfrac{1}{4}m^2\tilde{y}^2 - \pi g^2\tilde{y}^4 - \tfrac{2}{3}\sqrt{\pi}\lambda\tilde{y}^3. \tag{A1}$$

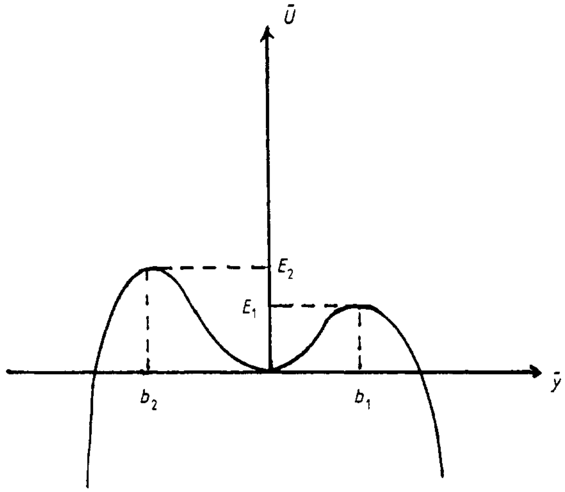

Figure 1.

Then the energy integral is

$$E=\tfrac{1}{2}(\mathrm{d}\tilde{y}/\mathrm{d}u)^2+\tfrac{1}{4}m^2\tilde{y}^2-\pi g^2\tilde{y}^4-\tfrac{2}{3}\sqrt{\pi}\lambda\tilde{y}^3 \tag{A2}$$

where E is the integral constant (total energy). Introducing

$$\tilde{y}=f+b \tag{A3}$$

and b determined by the condition

$$-\pi g^2b^4-\tfrac{1}{3}(4\pi)^{1/2}\lambda b^3+\tfrac{1}{4}m^2b^2-E=0 \tag{A4}$$

we can recast equation (A2) as

$$\left(\frac{\mathrm{d}\ln f}{\mathrm{d}\tau}\right)^2=1-\frac{D}{A}f+\frac{B}{A}f^2+\frac{K}{Af} \tag{A5}$$

where

$$\begin{aligned}
A&=12\pi g^2b^2+4\sqrt{\pi}\lambda b-\tfrac{1}{2}m^2\\
B&=2\pi g^2\\
D&=-8\pi g^2b-\tfrac{4}{3}\sqrt{\pi}\lambda\\
K&=8\pi g^2b^3+4\sqrt{\pi}\lambda b^2-m^2b\\
\tau&=\sqrt{A}u.
\end{aligned} \tag{A6}$$

In order to give a specific soliton solution, let us choose

$$K=0=\mathrm{d}\tilde{U}/\mathrm{d}\tilde{y}|_{\tilde{y}=b} \qquad E=\tilde{U}(b) \tag{A7}$$

where

$$b=b_{\frac{1}{2}}=\frac{-\lambda\sqrt{\pi}\pm(\pi\lambda^2+2\pi g^2m^2)^{1/2}}{4\pi g^2}. \tag{A8}$$

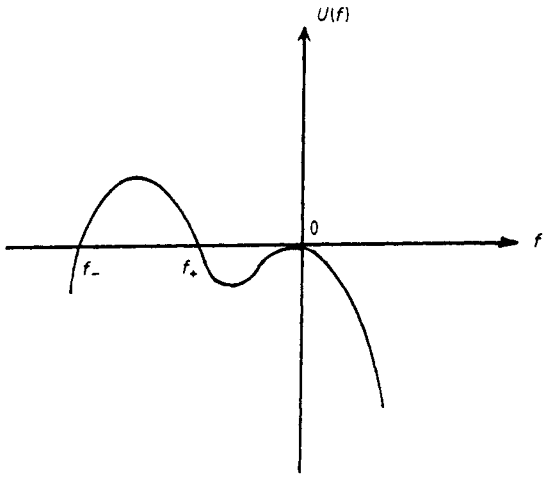

Figure 2.

The equation then becomes

$$\left(\frac{\mathrm{d}f}{\mathrm{d}\tau}\right)^2-\frac{B}{A}f^4+\frac{D}{A}f^3-f^2=0$$
$$U(f)=\frac{1}{2}\left(-\frac{B}{A}f^4+\frac{D}{A}f^3-f^2\right) \tag{A9}$$

and the corresponding total energy $E'=0$.

It can easily be proved that the equation has solutions

$$f=\frac{2R\,\mathrm{e}^{\pm\tau}}{(D/2A+R\,\mathrm{e}^{\pm\tau})^2-B/A} \tag{A10}$$

where the positive and negative signs in the exponential correspond to $\tau>0$ and $\tau<0$ respectively. The integral constant R is chosen as

$$R=1/f_+-D/2A \tag{A11}$$

to guarantee the periodic boundary conditions, where

$$f_+=[D+(D^2-4AB)^{1/2}]/2B \tag{A12}$$

is the turning point of the potential (figure 2).

From figure 2 we see that the particle with total energy zero can move only in the region $[f_+,0]$; otherwise it will go to infinity when $\tau\to\infty$. If the particle starts from any point between f_+ and 0, its motion at $\tau>0$ and $\tau<0$ will not be symmetric. So we must have $f(\tau=0)=f_+$ and from this condition we get (A11).

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